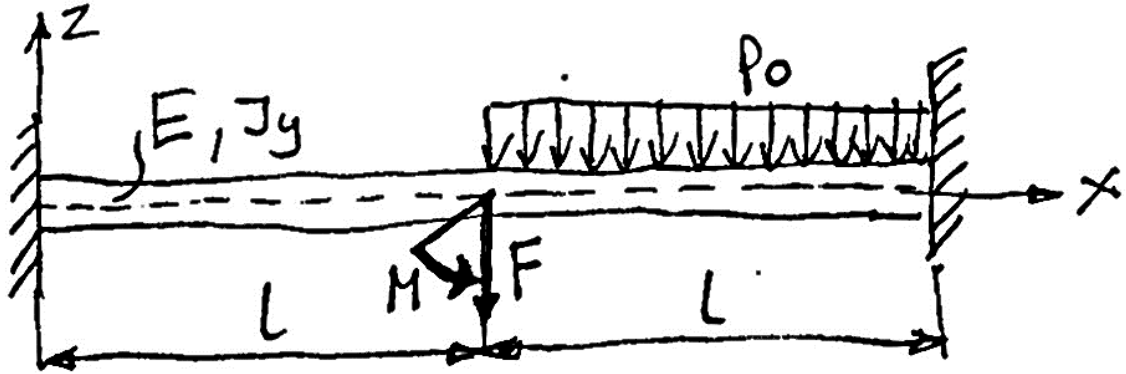
Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 9B. 1D beam element - examples

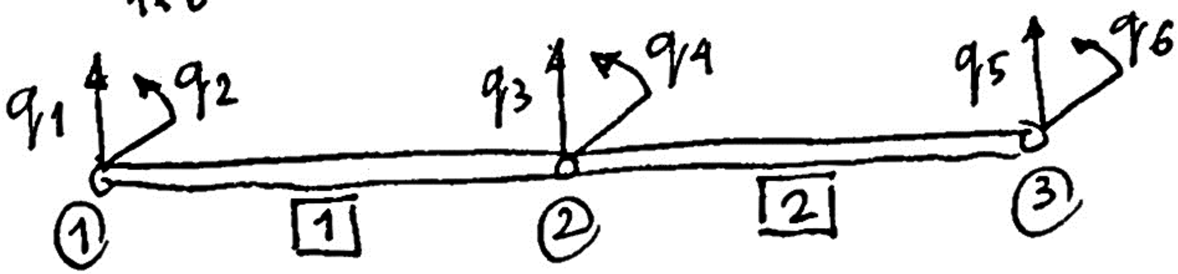
05.2025

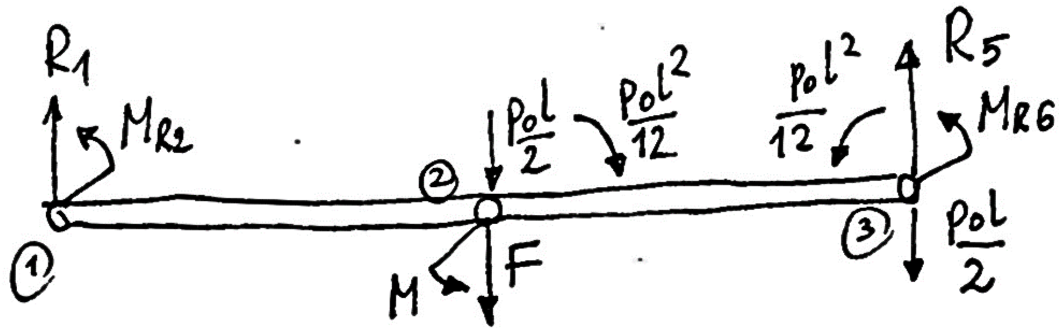
Example: Write a set of equations. Find nodal displacements, reactions and bending moment distribution. Use two beam elements.



$$\underset{1\times 6}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6 \rfloor$$

Nodal parameters





Nodal load

$$\underset{1\times 6}{\lfloor F \rfloor}^{n} = \lfloor R_1, M_{R2}, -F, M, R_5, M_{R6} \rfloor$$

Equivalent load

$$\underset{1\times 6}{\lfloor F \rfloor}^{e} = \underset{1\times 6}{\lfloor F \rfloor}^{*}_{1} + \underset{1\times 6}{\lfloor F \rfloor}^{*}_{2} = \lfloor \overbrace{F_{11}, F_{21}, F_{31}, F_{41}}^{=0}, 0, 0 \rfloor +$$

$$+ \lfloor 0, 0, F_{12}, F_{22}, F_{32}, F_{42} \rfloor =$$

$$= \lfloor 0+0, 0+0, 0-\frac{p_0 l}{2}, 0-\frac{p_0 l^2}{12}, 0-\frac{p_0 l}{2}, 0+\frac{p_0 l^2}{12} \rfloor$$

Global load

$$\underset{1\times 6}{\lfloor F \rfloor} = \underset{1\times 6}{\lfloor F \rfloor}^{n} + \underset{1\times 6}{\lfloor F \rfloor}^{e} =$$

$$= \lfloor R_1, M_{R2}, -F-\frac{p_0 l}{2}, M-\frac{p_0 l^2}{12}, R_5-\frac{p_0 l}{2}, M_{R6}+\frac{p_0 l^2}{12} \rfloor =$$

$$= \lfloor F_1, F_2, F_3, F_4, F_5, F_6 \rfloor$$

(N) (Nm) ...

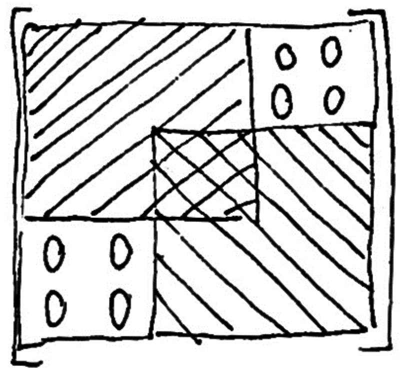
Stiffness matrices

$$\underset{4\times4}{[k]_1} = \underset{4\times4}{[k]_2} = \frac{2EJ_y}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

$$\underset{6\times6}{[k]_1^*} = \left[\begin{array}{c|cc} \underset{4\times4}{[k]_1} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}\right] \; ; \quad \underset{6\times6}{[k]_2^*} = \left[\begin{array}{c|c} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & [k]_2 \end{array}\right]$$

Global stiffness matrices

$$\underset{6\times6}{[K]} = \underset{6\times6}{[k]_1^*} + \underset{6\times6}{[k]_2^*} =$$



Set of equations + boundary conditions

$$\underset{6\times6}{[K]} \cdot \underset{6\times1}{\{q\}} = \underset{6\times1}{\{F\}} \; + \; \text{B.C.:}\; q_1=0,\; q_2=0,\; q_5=0,\; q_6=0$$

Solution

$$\cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{2EJ_y}{l^3} \begin{bmatrix} 6+6 & -3l+3l \\ -3l+3l & 2l^2+2l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

$$\frac{EJ_y}{l^3} \begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix} \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

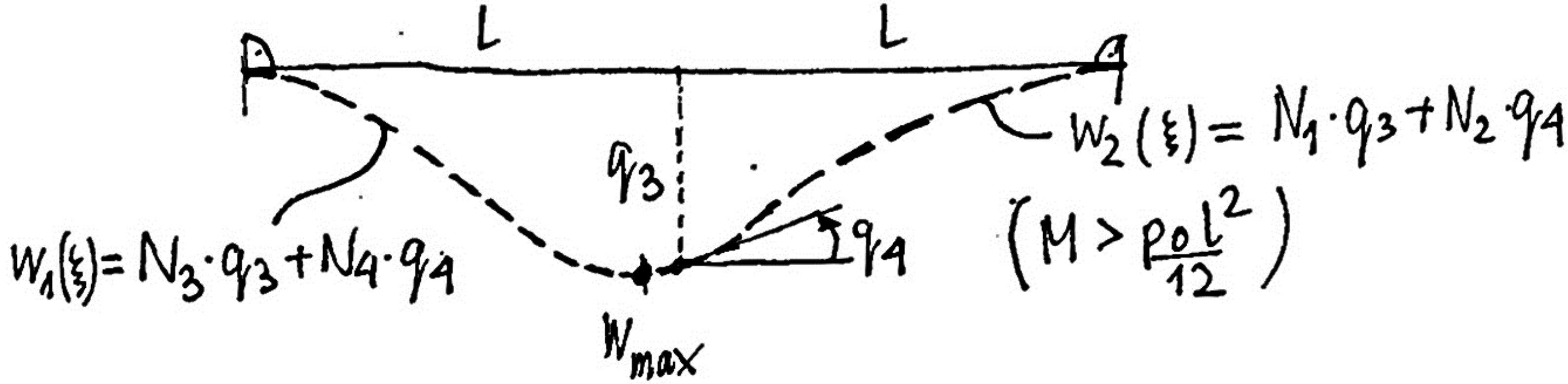
$$\frac{24EJ_y}{l^3} \cdot q_3 = F_3 \quad \Rightarrow \quad \boxed{q_3 = -\frac{\left(F + \frac{p_0 l}{2}\right) l^3}{24EJ_y}} \quad (mm)$$

$$\frac{8EJ_y}{l} q_4 = F_4 \quad \Rightarrow \quad \boxed{q_4 = \frac{\left(M - \frac{p_0 l^2}{12}\right) l}{8EJ_y}} \quad (rad)$$

Beam deflection

$$W_1(\xi) = \lfloor N \rfloor \cdot \{q\}_1 = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix}_1$$

$$W_2(\xi) = \lfloor N \rfloor \cdot \{q\}_2 = \lfloor N_1, N_2, N_3, N_4 \rfloor \cdot \begin{Bmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}_2$$



Reactions

$$\frac{2EJ_y}{l^3}\left\lfloor 6,\ 3l,\ -6,\ 3l,\ 0,\ 0 \right\rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_1 = R_1$$

$$R_1 = -\frac{12EJ_y}{l^3}\cdot\left(-\frac{(F+\frac{p_0 l}{2})l^3}{24EJ_y}\right) + \frac{6EJ_y}{l^2}\cdot\frac{(M-\frac{p_0 l^2}{12})l}{8EJ_y} =$$

$$= \frac{1}{2}\left(F+\frac{p_0 l}{2}\right) + \frac{3}{4l}\left(M-\frac{p_0 l^2}{12}\right) =$$

$$= \boxed{\underbrace{\frac{1}{2}F + \frac{3}{4l}\cdot M + \frac{3}{16}p_0 l}}$$

$$\frac{2EJ_y}{l^3}\left[3l,\ 2l^2,\ -3l,\ l^2,\ 0,\ 0\right]\cdot\begin{Bmatrix}0\\0\\q_3\\q_4\\0\\0\end{Bmatrix} = F_2 = M_{e2}$$

$$M_{e2} = -\frac{6EJ_y}{l^2}\cdot\left(-\frac{\left(F+\frac{p_0 l}{2}\right)l^3}{24EJ_y}\right) + \frac{2EJ_y}{l}\cdot\frac{\left(M-\frac{p_0 l^2}{12}\right)l}{8EJ_y} =$$

$$= +\frac{1}{4}\left(F+\frac{p_0 l}{2}\right)l + \frac{1}{4}\left(M-\frac{p_0 l^2}{12}\right) = \boxed{\underline{\frac{1}{4}Fl + \frac{1}{4}M + \frac{5}{48}p_0 l^2}}$$

$$\frac{2EJ_y}{l^3}\lfloor 0,\ 0,\ -6,\ -3l,\ 6,\ -3l \rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_5 = R_5 - \frac{p_0 l}{2}$$

$$R_5 = \frac{-12\,EJ_y}{l^3}\cdot\left(-\frac{\left(F+\frac{p_0 l}{2}\right)l^3}{24EJ_y}\right) - \frac{6EJ_y}{l^2}\,\frac{\left(M-\frac{p_0 l^2}{12}\right)l}{8EJ_y} + \frac{p_0 l}{2} =$$

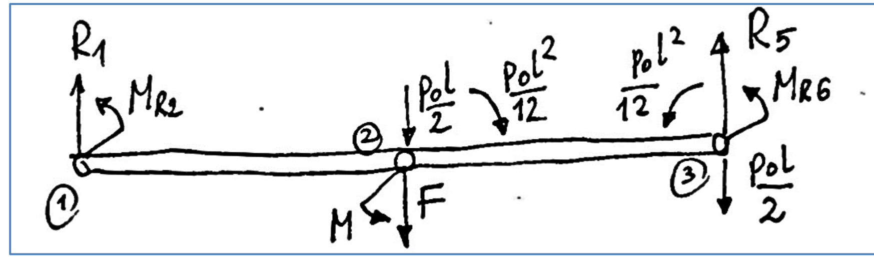
$$= \frac{1}{2}\left(F+\frac{p_0 l}{2}\right) - \frac{3}{4l}\left(M-\frac{p_0 l^2}{12}\right) + \frac{p_0 l}{2} \boxed{= \underbrace{\frac{1}{2}F - \frac{3}{4l}\cdot M + \frac{13}{16}p_0 l}}$$

$$\frac{2EJ_y}{l^3}\left\lfloor 0,\, 0,\, 3l,\, l^2,\, -3l,\, 2l^2 \right\rfloor \cdot \begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix} = F_6 = M_{R6} + \frac{p_0 l^2}{12}$$

$$M_{R6} = \frac{6EJ_y}{l^2}\left(-\frac{\left(F+\frac{p_0 l}{2}\right)l^3}{24EJ_y}\right) + \frac{2EJ_y}{l}\cdot\frac{\left(M-\frac{p_0 l^2}{12}\right)l}{8EJ_y} - \frac{p_0 l^2}{12} =$$

$$= -\frac{1}{4}\left(F+\frac{p_0 l}{2}\right)l + \frac{1}{4}\left(M - \frac{p_0 l^2}{12}\right) - \frac{p_0 l^2}{12} = \boxed{-\frac{1}{4}Fl + \frac{1}{4}M - \frac{11}{48}p_0 l^2}$$

Equilibrium check



$$\sum F_z = 0$$

$$R_1 - \frac{p_0 l}{2} - F - \frac{p_0 l}{2} + R_5 = 0$$

$$\frac{1}{2} F + \frac{3}{4l} \cdot M + \frac{3}{16} p_0 l - \frac{p_0 l}{2} - F - \frac{p_0 l}{2} + \frac{1}{2} F - \frac{3}{4l} \cdot M + \frac{13}{16} p_0 l = \underline{0}$$

$$\sum M_y^{x=0} = 0 \qquad +\circlearrowright$$

$$M_{R2} - \frac{p_0 l}{2} \cdot l + M - F \cdot l - \frac{p_0 l^2}{12} + \frac{p_0 l^2}{12} + R_5 \cdot 2l + M_{R6} - \frac{p_0 l}{2} \cdot 2l = 0$$

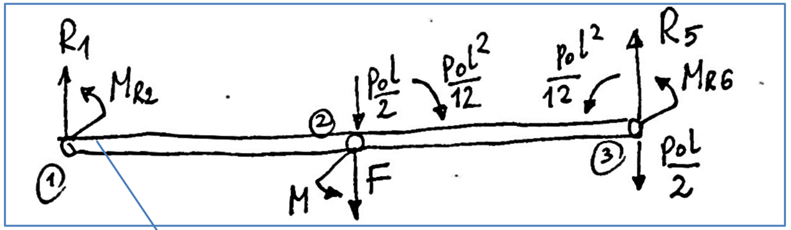
$$M_{R2} - \frac{3}{2} p_0 l^2 + M - F \cdot l + R_5 \cdot 2l + M_{R6} =$$

$$= \frac{1}{4} F l + \frac{1}{4} M + \frac{5}{48} p_0 l^2 - \frac{3}{2} p_0 l^2 + M - F \cdot l + F \cdot l - \frac{3}{2} M + \frac{13}{8} p_0 l^2 +$$

$$- \frac{1}{4} F l + \frac{1}{4} M - \frac{11}{48} p_0 l^2 = \frac{5}{48} p_0 l^2 - \frac{3}{2} p_0 l^2 + \frac{13}{8} p_0 l^2 - \frac{11}{48} p_0 l^2 =$$

$$= \frac{5 - 3 \cdot 24 + 13 \cdot 6 - 11}{48} p_0 l^2 = \frac{5 - 72 + 78 - 11}{48} p_0 l^2 = \underline{0}$$

Bending moment



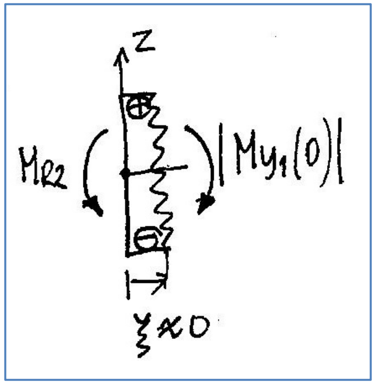
$\boxed{1}$: $M_{y_1}(\xi) = EJ_y W_1'' = EJ_y \lfloor N'' \rfloor_{1\times4} \cdot \{q\}_{4\times1,\,1} =$

$$= EJ_y \lfloor N_1'', N_2'', N_3'', N_4'' \rfloor \cdot \begin{Bmatrix}0\\0\\q_3\\q_4\end{Bmatrix}_1 = EJ_y\left(N_3''\cdot q_3 + N_4''\cdot q_4\right) =$$

$$= EJ_y\left(\left(\frac{6}{l^2}-\frac{12}{l^3}\xi\right)q_3 + \left(-\frac{2}{l}+\frac{6}{l^2}\cdot\xi\right)\cdot q_4\right)$$

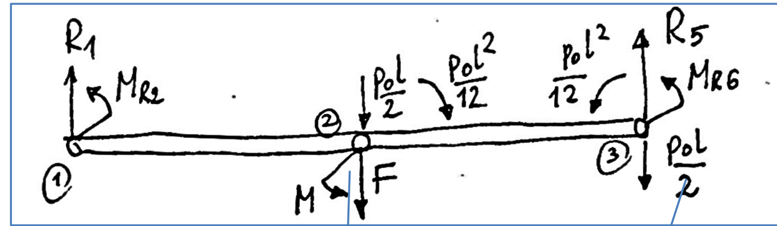
$$M_{y_1}(0) = EJ_y\left(\frac{6q_3}{l^2}-\frac{2q_4}{l}\right) = -\frac{6\left(F+\frac{p_0 l}{2}\right)l^3}{24l^2} - \frac{2\left(M-\frac{p_0 l^2}{12}\right)l}{8l} =$$

$$\boxed{= -\frac{1}{4}\left(Fl+\frac{5}{12}p_0 l^2+M\right)} \qquad \left(= -M_{R2}\right)$$



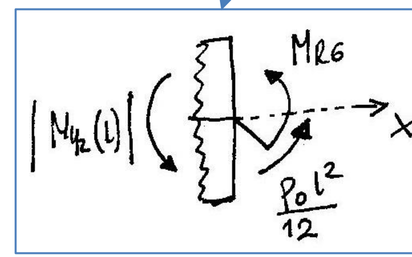
$$M_{y_1}(l) = EJ_y\left(\left(\frac{6}{l^2}-\frac{12}{l^2}\right)\cdot q_3 + \left(-\frac{2}{l}+\frac{6}{l}\right)q_4\right) = EJ_y\left(-\frac{6}{l^2}\cdot q_3+\frac{4}{l}q_4\right)=$$

$$= \frac{6\left(F+\frac{p_0 l}{2}\right)l^3}{24l^2} + \frac{4\left(M-\frac{p_0 l^2}{12}\right)l}{8l} = \boxed{\frac{1}{4}\left(F\cdot l+\frac{1}{3}p_0 l^2+2M\right)}$$



$$\boxed{2}: \quad M_{y_2}(\xi) = EJ_y W_2'' = EJ_y \lfloor N_1'', N_2'', N_3'', N_4'' \rfloor \cdot \begin{Bmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{Bmatrix}_2 =$$

$$= EJ_y \left(N_1'' \cdot q_3 + N_2'' \cdot q_4\right) = EJ_y \left(\left(-\frac{6}{l^2} + \frac{12}{l^3}\xi\right) q_3 + \left(-\frac{4}{l} + \frac{6}{l^2}\xi\right) \cdot q_4\right)$$



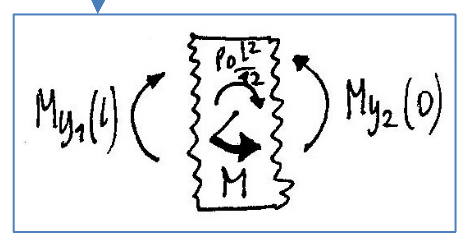
$$M_{y_2}(0) = EJ_y \left(-\frac{6}{l^2} q_3 - \frac{4}{l} q_4\right) = \frac{6\left(F + \frac{p_0 l}{2}\right) l^3}{24 l^2} - \frac{4\left(M - \frac{p_0 l^2}{12}\right) l}{8 l} =$$

$$= \frac{1}{4}\left(Fl + \frac{2}{3} p_0 l^2 - 2M\right)$$

$$M_{y_2}(l) = EJ_y \left(\frac{6}{l^2} q_3 + \frac{2}{l} q_4\right) = -\frac{6\left(F + \frac{p_0 l}{2}\right) l^3}{24 l^2} + \frac{2\left(M - \frac{p_0 l^2}{12}\right) l}{8 l} =$$

$$= -\frac{1}{4}\left(F \cdot l + \frac{7}{12} p_0 l^2 - M\right)$$

$$\left(= M_{R6} + \frac{p_0 l^2}{12}\right)$$



$$M_{y_1}(l) - M_{y_2}(0) = \frac{1}{4}\left(F \cdot l + \frac{1}{3} p_0 l^2 + 2M\right) - \frac{1}{4}\left(F \cdot l + \frac{2}{3} p_0 l^2 - 2M\right) =$$

$$= M - \frac{p_0 l^2}{12}$$

Let's assume:

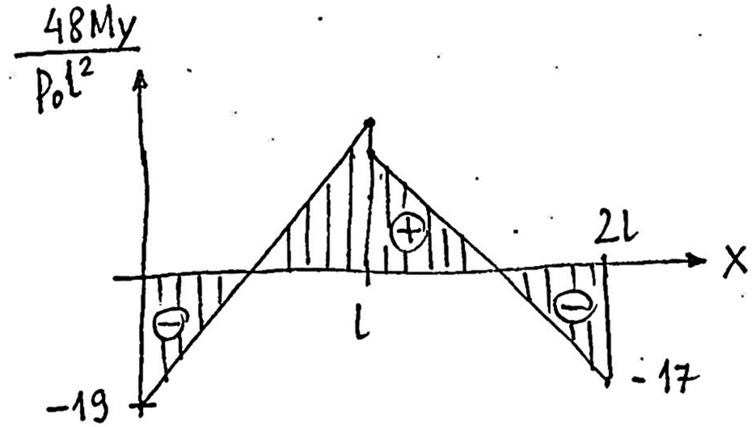
$$F = p_0 \cdot l \quad , \quad M = \frac{p_0 l^2}{6}$$

$$M_{y_1}(0) = -\frac{1}{4}\left(p_0 l^2 + \frac{5}{12} p_0 l^2 + \frac{p_0 l^2}{6}\right) = -\frac{19}{48} p_0 l^2$$

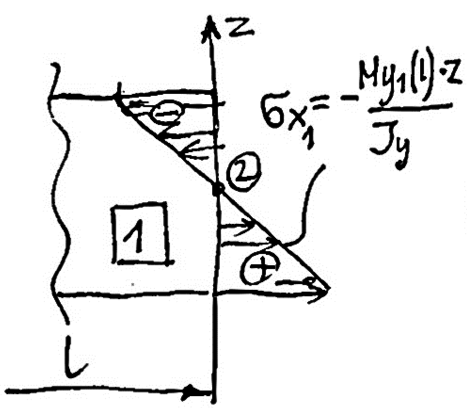
$$M_{y_1}(l) = \frac{1}{4}\left(p_0 l^2 + \frac{1}{3} p_0 l^2 + \frac{1}{3} p_0 l^2\right) = \frac{20}{48} p_0 l^2$$

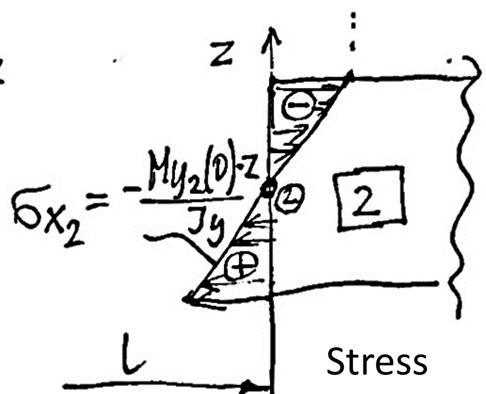
$$M_{y_2}(0) = \frac{1}{4}\left(p_0 l^2 + \frac{2}{3} p_0 l^2 - \frac{1}{3} p_0 l^2\right) = \frac{16}{48} p_0 l^2$$

$$M_{y_2}(l) = -\frac{1}{4}\left(p_0 l^2 + \frac{7}{12} p_0 l^2 - \frac{p_0 l^2}{6}\right) = -\frac{17}{48} p_0 l^2$$



Normal stress





Stress discontinuity